ETH
Eidgenössische Technische Hochschule Zürich
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Master Thesis/ Semester Project

Path Decomposition of Random Directed Graphs

A directed graph D (or digraph for short) is a graph with a vertex set $V(D)$ and edge set $E(D) \subseteq V^2$. Instead of the usual degree notion, we consider the *in-* and *out-degree* of a vertex: $\mathrm{d}^-(v)$ counts the number of edges going *into* v and $\mathrm{d}^+(v)$ counts the number of edges *leaving* v.

A *path decomposition* of a digraph D is a family of paths such that every edge of D is contained in exactly one path of the family. The path number $\mathrm{pn}(D)$ is the minimal number of paths necessary for a path decomposition of D.

Upperbounds: It was conjectured by Alsbach and Pullmann [1], and later proved by O'Brien [5] that the number of path required to decompose any digraph is at most $n^2/4$. This bound is tight (consider the complete bipartite graph $K_{n/2,n/2}$ in which all edges are oriented in the same direction), but very far from the truth for many graphs.

For various families of graphs, one can prove an significantly smaller upperbound: for instance, it was conjectured by Bollobás and Scott [2] that $O(n)$ many paths are sufficient for any n-vertex *Eulerian* digraph (i.e. $\mathrm{d}^-(v) = \mathrm{d}^+(v)$ for all vertices v) and in a recent work [4], we show that this is true up to a factor $\log \overline{d}$, where $\overline{d}$ denotes the *average degree* of the digraph.

Lowerbounds: Regarding lower bounds one easily checks that for any graph D, $\mathrm{pn}(D) \geq \frac{|E(D)|}{n-1}$, as each path can cover at most $n-1$ many edges. After thinking for a bit, one can find another very interesting lower bound: we define the *excess* at a vertex v as $\mathrm{ex}(v) := \mathrm{d}^+(v) - \mathrm{d}^-(v)$ and the total excess of the graph D as

$$\mathrm{ex}(D) := \frac{1}{2} \sum_{v \in V(D)} |\mathrm{ex}(v)|.$$

For any path decomposition and any given vertex $v \in V(D)$, we need to have at least $|\mathrm{ex}(v)|$ paths that start or end in v (depending on whether the excess is positive or negative), and this clearly implies that

$$\mathrm{pn}(D) \geq \mathrm{ex}(D).$$

A graph is called *consistent* if there is equality, i.e. if $\mathrm{pn}(D) = \mathrm{ex}(D)$. One could hope that all graphs are consistent (as it would give a simple method of computing the path number of all graphs), but this is not the case since Eulerian graphs have excess 0 but require at least one (and possibly many) path(s) to be decomposed.

Yet, in a recent work, Espuny Díaz, Patel and Stroh [3] showed that the random digraph $D_{n,p}$ is consistent with probability $1 - o(1)$ if

$$\frac{\log^4 n}{n^{1/3}} \leq p \leq 1 - \frac{\log^{5/2} n}{n^{1/5}}.$$

Here, a *random digraph* $D \sim D_{n,p}$ is sampled by adding each of the possible $n(n-1)$ edges independently with probability p (similar to the binomial random graph model $G_{n,p}$).

Goal of the Project In this project you will explore how $\mathrm{pn}(D)$ behaves in $D_{n,p}$ in other regimes of p. Your goals will be:

- Familiarize yourself with state-of-the-art exploration methods.
- Apply known methods to $D_{n,p}$ and see where the limitations are.
- Find general upper bounds for $\mathrm{pn}(D_{n,p})$. Evaluate if these are best possible for given regimes of p.
- Determine for which p we have that $D_{n,p}$ is consistent.

More information and grading scheme can be found on:
`https://www.cadmo.ethz.ch/education/thesis/guidelines.html`

Prerequisites: Random graphs - ideally you took our course 'Randomised Algorithms an Probabilistic Methods'

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References

[1] B. R. Alspach and N. J. Pullman. Path decompositions of digraphs. *Bulletin of the Australian Mathematical Society*, 10(3):421–427, 1974.

[2] B. Bollobás and A. D. Scott. A proof of a conjecture of bondy concerning paths in weighted digraphs. *journal of combinatorial theory, Series B*, 66(2):283–292, 1996.

[3] A. E. Díaz, V. Patel, and F. Stroh. Path decompositions of random directed graphs. *arXiv preprint arXiv:2109.13565*, 2021.

[4] C. Knierim, M. Larcher, A. Martinsson, and A. Noever. Long cycles, heavy cycles and cycle decompositions in digraphs. *Journal of Combinatorial Theory, Series B*, 148:125–148, 2021.

[5] R. C. O'Brien. An upper bound on the path number of a digraph. *Journal of Combinatorial Theory, Series B*, 22(2):168–174, 1977.